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Société et culture
Québec

Rapport de recherche

PROGRAMME ACTIONS CONCERTÉES

La réussite en mathématiques au secondaire commence à la maternelle: Synthèse des connaissances sur les pratiques d'enseignement des mathématiques efficaces à la maternelle et au primaire pour réussir l'algèbre du secondaire?

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Introduction

For many of us, the word *algebra* brings up words such as equations, unknowns, variables, and functions—the world of symbols and rules about how to deal with those symbols. Some would say it was the best part of their experience of school mathematics, because it came down to strictly following a set of rules and get to the answer. Others would argue that it was the worst of their experience, because all those symbols were devoid of meaning. Yet, all would most probably remember when, in secondary school, they first met algebra. Traditionally, algebra is taught in a way very different from one students used to in arithmetic. This drastic change in the way of doing mathematics may have caused many to see algebra as an obstacle.

Problems associated with the transition from elementary school arithmetic to secondary school algebra have been gaining increase attention in recent years (e.g., Bednarz & Janvier, 1993; Kieran, 1989, 2007). To respond to the arithmetic-algebra gap, research in the past two decades focused on exploring the possibilities of introducing elements of algebraic thinking at a much earlier age. Researchers (ex. Brizuela et Shliman, 2004; Cai et al., 2011), collectively, agree that the development of algebraic thinking in primary and preschool students has great potential in eliminating or significantly reducing difficulties in learning algebra at the high school level. However, from our research, it follows that doing "algebraic tasks" is less important than doing any task "algebraically". It means that while doing arithmetic tasks, students can and should think algebraically. If students develop this way of thinking from early grades, there will no need to change it when they arrive at the secondary level and thus the gap between arithmetic will be reduced.

The recommendations in this guide are well in line with the Quebec elementary mathematics curriculum. While the pedagogical suggestions we offer here do not involve a drastic change in the content of tasks, they do require a change in the way activities are carried out in the classroom. We offer a refined way of doing arithmetic that is characterized by prioritizing fundamental and general mathematical ideas and by a better balance between the teaching of arithmetic operations on the one hand and structures and mathematical relationships on the other hand. We also stress the important role of modeling, and mathematical classroom discussions. Thus, we first explain what early algebraic thinking is and then present some mathematical activities that can potentially allow for the development of algebraic thinking in young students.

Early algebraic thinking

The term *algebraic thinking* refers to the practice of generalizing different kinds of situations looking for their essential properties and structure, and to the skill of communicating these general ideas in different ways (symbolic, verbal, schematic etc.). Generalization is a human-specific thinking tool. As such, we concur with Mason (2018) who asserts that it is never too early to think algebraically. Even a 3-year-old child can play with a scale to weigh sand or toys, to observe the difference, or to try to match the weight of his doll with that of a cub. The 3-year-old can thus generalize the idea of equivalence of weight (see for e.g., Davydov, 1982; Wallace et al., 2010). There are many other contexts that allow young children to immerse themselves in the world of quantities, logic, and generalization without numbers—important building blocks in developing algebraic thinking.

Based on a rigorous synthesis of research, we identified five important components of early algebraic thinking. In each component we offer a set of possible tasks that can contribute to the development of the respective component of algebraic thinking.

Component 1. *Generalization of patterns*

Generalization of patterns includes the skill to observe patterns created in different contexts (using varied materials), to recognize their structure (what is repeated from one element to another and what changes), to verbally describe or model the generating principle of the pattern, to continue the pattern in accordance with the generating principle, to construct the elements according to their position in the pattern set, and to create other patterns according to the model or the description of the generator principle.

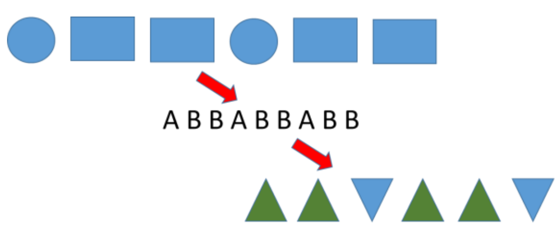
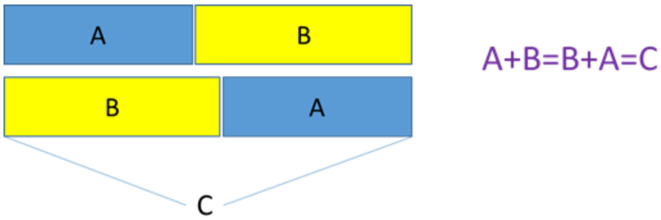


Figure 1 Modeling and recreating patterns (Wijns, 2019).

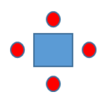
Component 2. *Relational thinking*

This component of algebraic thought refers to developing a skill of seeing a situation or problem as a set of relations between quantities. It includes the ability to recognize relationships (example: a quantity is composed of two other quantities, or 5 = 3 + 2, or x+2=5), describe them verbally, and represent them by way of modeling. It also requires knowledge of the links between quantitative relationships and arithmetic operations (see for e.g., Polotskaia & Savard, 2018). The component of relational thinking is essential for modeling and problem solving.



Component 3. ***Functional thinking***

The functional thinking component refers to the appreciation of direct (functional) relationship between two co-varying quantities (see for e.g., Blanton et al., 2015). The functional relationship allows a bidirectional move between the co-varying quantities. For example, in the following task, one should be able to establish direct relationship between the number of tables and the number of seats: find the number of tables needed for 16 (or 160) people or find the number of people that can be seated around 2 (200) tables. $T=(S-2)\div 2$; $S=T\times 2+2$[1]. To find the number of seats around 4 tables, one can do $4\times 2+2$.

Tables	**Figure**	**Seats**
1		4
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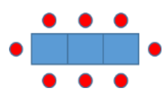
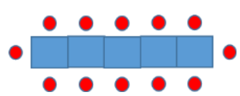
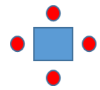
		8
?		16
		

Component 4. ***Recursive thinking***

Recursive thinking allows us to link consecutive elements of a sequence or successive performance of an operation. For example, consider the same task of the tables and the seats, one can observe that adding

[1] There are many other ways to express these direct relationships.

one table results in obtaining two more seats: $T_{(n+1)}=T_n+2$; To find the number of seats around 4 tables, we can do 4+2+2+2.

Tables	**Figure**	**Sits**
1		4
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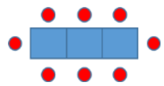
3		?
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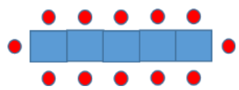
5		
…	…	…

The main difference between *recursive thinking* and *functional thinking* is that recursive thinking relates two consecutive terms (their positions in the sequence are not considered); *functional thinking*, on the other hand, relates the term with its position in the sequence. As a result, the radius of action of the recursive thinking is one step ahead, while functional thinking allows going forward or back in an arbitrary number of steps.

Component 5. *Modeling*

Modeling refers to the skill of expressing essential properties of an object or situation in some way different form how it is already expressed. In the above example of patterns in the section *Generalization of Patterns*, the sequence of letters (ABBABB) expresses a model of a general mechanism generating the pattern. In the section of *Relational Thinking*, we show colored bands composing a new length for which the expression C=A+B can be a model of the situation. These combined bands can serve as a model of a numerical situation such as 2+3=5. It is important to notice that this model is a formalization of **essential relationships** between the quantities involved in the situation. Thus, *modeling* is intertwined with *relational thinking* and *functional thinking*. An essential aspect of *modeling* is a reformulation of a given situation in a **different** (from the one given in the situation) representational system. Such reformulation requires an identification of the essential components and a deep understanding of the relations between them. Letter notation is one of many ways to model situations. Research suggests that letter notation is widely applicable and accessible for students as early as 6-year olds (see for e.g., Davydov, 1982; Lee, 2006)

Collectively, these thinking skills relate to mathematical content of the school curriculum although they are not explicitly mentioned in the documents. For example, in a situation involving several boxes packed with the same number of apples each, it would be advantageous for the learner to understand that there is a general relationship between the number of boxes and the total number of apples. Without such understanding, the construction of the meaning of multiplication and division operations is unimaginable. In fact, the relationship between the number of boxes and that of apples is an instance of **functional relationship**. According to the empirical research (Daydov, 1982; Pasnak et al., 2009; Smith & Thompson, 2008), the above-described ways of thinking can help students develop their arithmetic knowledge more efficiently and sustain these skills for problem solving.

At an even more fundamental—and foundational—level, researchers in mathematics education highlight several basic principles that guide human experiences within the quantitative world. We refer to these principles as mathematical roots, because they underpin the development of mathematical and algebraic thought. Children rely naturally on these principles. In fact, research has demonstrated that children involve these principles in a tacit, not readily observable manner and that they do so in multiple, diverse occasions. The purpose of specifically designed activities for kindergarten teachers would be to allow for each child to consciously appreciate and explicitly harness these basic principles so that she attribute an appropriate and clear sense the adult's communication about the mathematical idea associated. In the consulted literature, scholars mention the following fundamental principles:

> *Number and quantity conservation* refers to the understanding that the numerosity of a set doesn't depend on the positioning of the elements, or that the amount of substance (for example, clay) does not change if we change its shape (e.g., Pasnak et al., 2009).
>
> *Oddity principle* refers to the process of identifying a feature based on which a certain element can be discriminated from a set. This process requires an abstraction of the object along several possible features and an analysis / comparison of that feature with the feature of other elements of the set (e.g., Pasnak et al., 2009).
>
> *Seriation principle* refers to recognition of the principle underlying the order in which the objects are organized in a sequence. If one needs to insert an object, she must refer to this principle

(according to the characteristics of the object) to find its place in the sequence (e.g., Pasnak et al., 2009).

Commutativity of addition refers to the understanding that the result of addition (combination of amounts) is independent from the order in which the amounts were added (e.g., Blanton et al., 2015).

Additive identity refers to the understanding that adding or removing nothing (zero, 0) does not change the initial quantity (e.g., Blanton et al., 2015).

Addition-subtraction inverse principle (doing/undoing) refers to the understanding that the initial amount will not change if we add another amount A and, then, remove the same amount A (e.g., Lai et al., 2008).

Surely, there are more such principles to discover and develop. Research (for e.g., Pasnak et al., 2009) has shown that explicitly learning about such general ideas helps students (including at-risk students) to learn numeracy and literacy more easily and effectively.

All researchers argue that the development of algebraic thinking requires a long-term, carefully designed instruction through well-structured activities along with long-term, continuous follow-up on the progress of students' algebraic thinking.

A word about the use of letters

Increasingly growing research demonstrates that students' difficulty in interpreting letters in algebra is rather artificial and that such difficulties are repeatedly attributed to a curriculum that dictates the use of letters in mathematics only at the secondary level. In cases where letters were used at Grade 1 (see for e.g., Lee, 2006) to designate quantities (known or unknown), students demonstrated such difficulties, but for the **first 3 hours** of the lesson only! Studies have shown that there are ways to introduce the use of letters at any time in primary school without creating a drastic conflict with previous learning (see for e.g., Freiman et al., 2017).

We would like to highlight the difference between a letter being used to represent a quantity and the notion of variable. The meaning given to a letter differs from situation to situation. For example, if we designate the number of chickens in a courtyard by the letter ***c*** this will stand for an unknown, but fixed—according to the given situation—quantity. Thus, a letter in a mathematical context may sometimes designate a known or unknown **constant** quantity. While a variable, also labelled by a letter, is a quantity that **varies**, according to the meaning of the situation. For example, the number of chicken in the courtyard **vary during the day**.

Next, we present a brief description on how to render an activity as an algebraic one, while using letters or other modeling tools.

How to transform an arithmetic activity into an algebraic one

In general, the construction of algebraic thought begins in a situation or problem formulated in a context that is familiar to the student. Such situations or problems may involve physical objects referring to the objects' measurable or quantifiable properties. At a later stage, such situations can be framed within word problems or other mathematical contexts.

The task should invite the student to analyze relationships between given quantities and model these relationships using a representational system that is different from the given one—that is, the student mobilizes representation of the relationships between the given quantities from, for example, a verbal description to a schematic one. The development of algebraic thought begins with the situation that is being *translated* with the objective of identifying generalities and relations. This translation transforms the situation—and, in turn, the student's understanding of the situation—into a model. Thus, a general solution to the problem can be constructed first within the context of model.

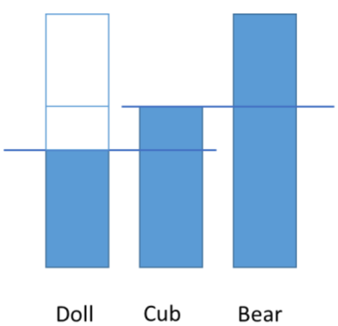
The general solution can then be interpreted in the initial context of the problem to make sense of the result. The context-model-context process is a crucial element of mathematical learning to ensure the gradual development of algebraic thought.

Thus, the main purpose of any mathematical activity is to highlight the general structure of a specific situation and use this structure to construct a general solution(s).

As we mentioned earlier, even at the age of 3, a child can experiment with differences between quantities by means of observation. The child can, for example, play with a set of scales (balance) to weigh sand or toys, or to try to match the weight of an object with that of different amounts of water. This can lead to a conversation about the general properties involved, such as the doll weighs **less than** the cub and **therefore** the cub weigh **more than** the doll. But if the doll eats a candy and gets bigger, it weighs **the same as** the cub. Later on, the teacher can use letters (or pictures) to introduce children to the concept of modeling (D<C). The teacher can even go further and ask what may happen if the doll and the cub eat a candy (K) each. (D+K ? C+K). Or what will happen if the doll weighs the same as the cub, but then the doll eats a big tart (T) and the cub eats a small cookie (K)? (D=C; T>K; D+T ? C+K). As the teacher facilitates this activity and allows the children to come up with different scenarios and their respective letter representations, the children are gradually introduced to appropriate vocabulary and deepen their understanding of this relationship between quantities. The weighing activities, if properly structured, will help students develop a deep understanding of the notion of equivalence and learn general properties of addition and subtraction. Thus, the experience with physical objects, and later with equivalence of numbers, lays the foundation for work with the notions of equations and inequalities.

In the quest for a general result, students' continual engagement with and practice in conjecturing, discussion, and proof, which are essential supporting elements for the development of thinking. Formulating conjectures requires envisioning generally valid properties and linking them to features explicitly identified in the situation. Through skilfully conducted discussions, the teacher can support students as they construct proofs and arguments for their statements. We cannot emphasize enough the fundamental role that making arguments and conjecturing play in any mathematical activity. In this process, the role of teacher is crucial. Any activity should promote children's formulating conjectures and stimulate discussion involving logical arguments and proofs.

For example, at some point in working with balance, the teacher can ask the children to "prove" a conjecture logically before verifying it on the real balance. If the doll weighs less than the cub and the cub weighs less than the bear, how can we be sure that the doll weighs less than the Bear without weighing them? Children can model the situation using bars as follows:



They can explain: Because the cub weighs more, we need to add something to the doll to create a balance. To balance this with the bear, we need to add some more weight, because the bear weighs even more than the cub. Thus, in order to obtain a balanced weight between the doll and the bear, we need to add a lot to the doll, which means that the doll weighs less than the bear.

Tasks and activities that allow children to advance several conjectures, and to revise and search for arguments to justify their validity of their conjectures not only offer a rich learning experience for students but also support the development of algebraic thinking. Such conditions stimulate the formation of a certain habit of mind where students investigate, formulate and revise statements. They also develop an appropriate understanding of how mathematics is done, including the fact that—more often than not—mathematical activities require a long time to figure out, careful thinking, and argumentation and might not have immediate nor definitive answers.

Speaking about arithmetic tasks, we should carefully consider the relationships between quantities and the meaning of arithmetic operations. In the current primary curriculum, the emphasis is on the study of arithmetic operations—mainly, from a procedural point of view. However, each arithmetic operation does not only represent a calculation instruction, but also a relation between the quantities concerned. What brings together arithmetic and algebra is the study of relationships between quantities. Arithmetic is more about in the methods of calculating quantities expressed by numbers, while algebra studies the relations between quantities and their properties. Let us explore a case of a simple arithmetic word problem through algebraic thinking point of view.

> *Marta has 2 oranges more than Olga. How many oranges does Olga have, if Marta has 5 oranges?*

From the arithmetic point of view, this is a problem to be solved by using the subtraction operation: 5-2=3. Very often the "official goal" of such tasks is to learn about addition and subtraction and to practice mental calculation. In order to bring more algebraic sense to the task, a teacher may ask the students to think about the first sentence only and to imagine other different concrete situations.

> *Marta has 2 oranges more than Olga.*

Is it possible that Olga has 2 oranges and Marta has 1? Why? Can it be that Olga has **no oranges at all**? What **can** possibly be the number of Olga's oranges and Marta's oranges? Can we model this situation for **all possible number** of oranges for Olga and Marta? Can we express this situation by using letters? What do these letters mean? In **any of such situations**, if we know the number of Olga's oranges, how can we find the number of Marta's oranges? And if we know the number of Marta's oranges, how can we find the number of Olga's oranges? Can we imagine other similar situations when two quantities are compared? If we know two compared quantities, how can we describe the situation in terms of comparison? In **any of such situations**, how can we act to find the difference? Can we show it in the previously constructed model or by using letter expressions?

These questions reveal the profound mathematical background of "simple" arithmetic problems. We should never underestimate their potential in developing algebraic thinking. Equally important, we should never underestimate students' potential in learning mathematics if appropriate conditions are offered.

In what follows, we present some typical examples of tasks, proposed by researchers, to support algebraic thinking development at different learning levels.

Age 3 to 6 years old

Research suggests that it is never too early to start thinking algebraically. Aside from the scale games that we described in the introduction, researchers have experimented with several other contexts and concepts. Here are some examples.

Conservation of number or quantity (inspired by Pasnak et al., 2009)

The principle of number conservation is the basis of our understanding the notion of number. To help young children acquire this principle, they need to grapple with specially designed situations and the teacher needs to ask some special questions.

The child is presented with several rabbits and a number of carrots (about 6-8). The child is asked to determine if there are more rabbits, more carrots, or as many carrots as rabbits. To answer these questions, the child is asked to give **one** carrot to **each** rabbit. This term-to-term correspondence informs the child of the meaning of the words "more than," "less than," "as much as."

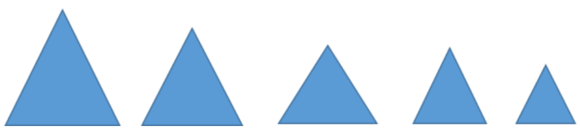
Rabbits and carrots are placed next to each other and it is confirmed with the child that there are as many rabbits as carrots, that each rabbit can have its carrot, and that there will no be carrots left. We then change the layout of rabbits so that the rabbits take up more space on the table. The question is repeated: determine whether there are more rabbits, more carrots or as many carrots as rabbits. The child who masters the principle of conservation will confirm equality without hesitation by explaining that no rabbits were added or removed. On the other hand, the child who does not master this principle will say that there are more rabbits. So, we repeat the question (Are there more rabbits?) And we suggest to the child to check if we can give a carrot to each rabbit.

We should not expect the child to master the conservation principle at once. The educator must regularly propose similar situations in various contexts (water in two identical containers, two pieces of modeling clay, etc.) so as to give the opportunity to the child to encounter the same principle in different contexts. For the activity to achieve its purpose, it must be ensured that an understandable and easily accessible method of verification (e.g., term-to-term) is available to the child.

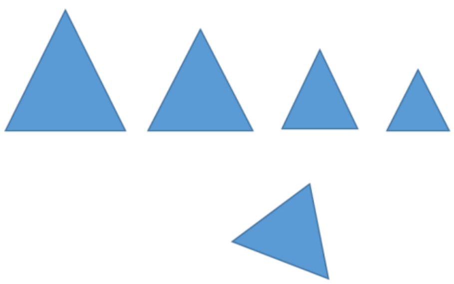
Principle of seriation (inspired by Pasnak et al., 2009)

The principle of seriation is not only at the base of algebraic thought, but it is also one of the pillars of mathematical reasoning.

To form a series, one can use various material such as sticks of different lengths, plane or solid figures of varied areas, colored pieces of paper with varying intensity of color, etc. Objects are arranged so that the quality in question (length, area, color intensity) changes from one object to another following an increasing or decreasing order.



The child is invited to observe the series and the teacher explains that the objects are arranged in order. The child is then asked to close their eyes, and the teacher removes an object from the series. Other objects are rearranged to "hide" the empty place.

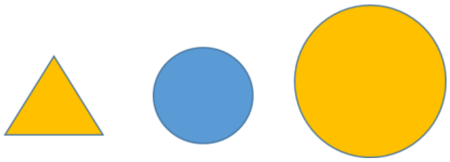


The child is asked to insert the object back "in its place" within the series. To make the game fairer, we can offer the child to hide an object and the teacher will need to find where the piece belongs. Obviously, the teacher may misplace the piece and give the child the opportunity to correct the mistake and explain why the piece does not belong where it was placed.

Oddity Principle (inspired by Pasnak et al., 2009)

The intruder game helps the child learn observation and analysis of similarity and difference. For the game, one can choose various contexts such as plane and solid geometric figures, words and syllables, food items, pictures of animals etc. You can play this game to enrich the vocabulary of the child and introduce basic concepts.

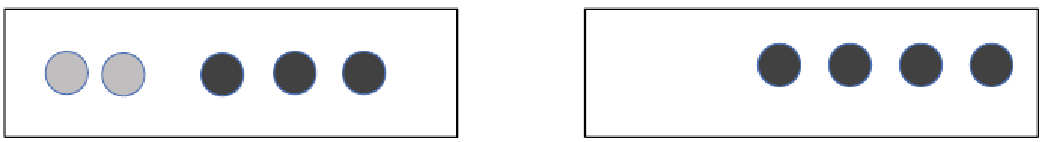
To play, we compose a set of three objects so that two of them have a common characteristic that the third object does not share. The child is asked to find out which of the objects does not belong. Objects can be combined to have two or three possible answers. Each time, the child is asked to explain the logic for the answer provided and the teacher takes the opportunity to help the child to express these thoughts in a correct mathematical language. In the example below, there are two circles and a triangle (the triangle is the odd one out), two orange figures and one blue (the small circle is the odd one out) and two small figures and one big object (the big circle is the odd one out). If the child has found a solution, it is important to invite them to think differently to find another solution.



Equivalence: The Card Game (Blanton et al., 2018)

Numbers are represented on strips of paper as a collection of tokens (for example, in Figure 1, the left rectangle shows gray and black tokens).

A. We compare two so two sets of collections of tokens.



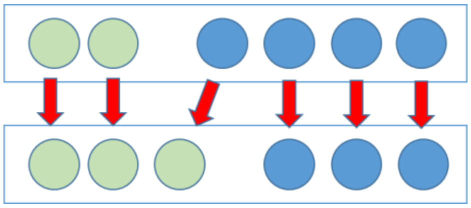
How many gray circles should you put on the 2nd card so that there is the same number of tokens on both cards?

The goal of the game is to determine the number of tokens to add on one of the cards so that both sets of collections are equal. In this example, we want the student to engage in solving an equality like a+b=_+c. We can obviously swap the cards and obtain a+_=b+c or use a single chip color to work the equalities of type a=b+c or b+c=a.

B. The teacher prepares pairs of cards. Each pair consists of two cards with the same number of tokens drawn on them.

Each child receives a card with gray and black tokens drawn on them and is asked to find a classmate who has a card with an equivalent number of tokens. From the mathematical point of view, we want the students to establish equalities of the type a+b=c+d. We could also begin the game with cards that only have black chips and thus offer the equality a=a.

In these two parts of the card game that were proposed by Blanton et al. (2018), the ultimate goal is to encourage students to translate the established equalities into mathematical terms. For example, when a child who has a card with 2 gray and 5 black tokens and another child who has 4 gray and 3 black tokens form a pair, the teacher can lead the children to write the established equality of 2 + 5 = 4 + 3. It is not about calculating 2 + 5 = 7, but establishing equality using term-to-term correspondence and expressing it mathematically.



Addition-subtraction inverse principle: The story of parking (Lai et al., 2008)

A black carpet and an opaque box are presented to the children. The teacher says that the black carpet is a covered parking and he slips some cars under the box (between 4 and 7 at random). The teacher then says we do not need to know how many cars there are. The teacher presents a story "This morning there were this many cars in the parking lot. In the afternoon more cars arrived and some cars left." The children are asked to take a good look at what is happening and determine "if, at the end of the day, there are more, less, or as many cars than there were in the morning."

The teacher then performs transformations in the following way: 1) placing a few cars (between 2 and 5) to the left of the carpet for 3 seconds then slides them under the box, OR 2) removing some cars from those initially placed under the box and sliding them to the right of the carpet, leaving them there for 3 seconds then removing them. The teacher then asks the children, "Are there now more, fewer, or as many cars in the parking lot as this morning?"

Variations of contexts (Baroody & Lai, 2007)

"This is Mickey Mouse's house (the box). In his house he has a plate with biscuits. The teacher places 19 chips to represent the cookies on the plate and then hides the plate with the box. "Minnie Mouse will remove cookies from or add cookies to the plate and then you will have to decide whether or not Mickey is happy with this change. Showing a smiling face (icon), the teacher says, "If Mickey has more cookies than before then he is happy." Showing a neutral face the teacher says that "If Mickey has the same number of cookies as before then he is not happy or sad." And, showing a sad face, the teacher says, "If Mickey has fewer cookies than before then he is sad."

Possible transformations:

• Add objects only.

• Remove objects only.

• Add a number of objects and remove less.

• Add a number of objects and remove more.

• Remove a number of objects and add less.

• Remove a number of objects and add more.

Note from authors (Baroody & Lai, 2007): The additions are always made with the left hand and the removal of cookies (objects) with their right hand.

Cycle 1 (6 to 8 years)

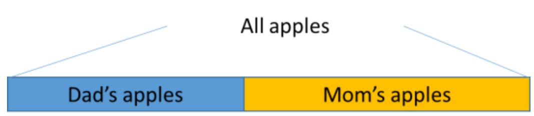
This cycle is usually characterized by intensive learning of number sense and properties of operations (addition and subtraction). It is therefore essential to complement arithmetic knowledge with relational knowledge.

Modeling problems (Polotskaia & Savard, 2018; Warren & Cooper, 2009).

The solving of simple written problems is an excellent opportunity to introduce and practice modeling. Before presenting the problem in its usual form using numbers, students are given a description of the situation without numbers.

Dad buys some apples. Mom buys some too. How many apples did they buy in total?

Students are asked to represent the situation in a way that it can be interpreted in a variety of data situations. In other words, it is not required to draw apples as discrete objects. We can, however, imagine the apples arranged as a line. These lines can be placed one as a continuation of the other to "see" the total quantity of apples that mom and dad bought.



From this model, we can find the operation that needs to by performed to solve the problem. In a different situation, if the number of mom's apples is the unknown, we can still refer to the model in order to propose an operation that needs to be performed. The strength of such modeling is that one can represent unknown quantities and discuss the solution strategies for "general" cases.

In the modeling activity, care must be taken that students understand the meaning attributed to the different elements of the model in the given context. The teacher may ask the students questions to reinforce the links of meaning between the text of the problem, the model, and the solution.

• The blue stripe represents what information that is given in the problem?

• Why did we put the stripes one after the other?

• How are all the apples purchased represented by the model?

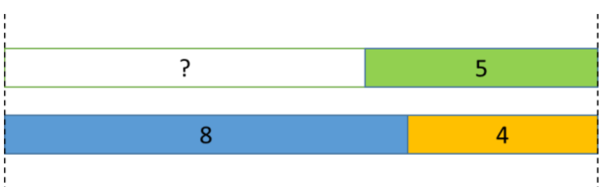
• Where is the number you found as an answer to the problem represented in the story? Where is it represented on the model?

Operations with missing numbers (Carpenter et al., 2005; Molina & Mason, 2009)

Students must complete the mathematical sentences by replacing □ with the correct number. To achieve this, one can model (see above) the expressions and or compose a story according to the meaning of the expression.

$8 + 4 = \square + 5$
$\square = 25 - 12$
$14 + \square = 13 + 4$
$13 - 7 = \square - 6$
$\square + 4 = 5 + 7$
$12 + 7 = 7 + \square$

$12 - 4 = 13 - \square$
$9 - 4 = \square - 3$
$\square - 6 = 15 - 7$
$14 - 9 = \square - 10$
$17 - \square = 18 - 8$



Part of the teacher's work consists of encouraging the students to verbalize their thought processes. This has a double purpose. One is to develop the students' mathematical communications skills; the other is for the students to explain their reasoning. For example, for the equation 8 + 4 = _ + 5 one student might proceed as adding first 8 with 4, then asking what number to add to 5 in order to obtain an equal result. Such a way of proceeding is procedural. Operations are carried out sequentially until the equation is reduced to a standard one. Another student can consider 5 as 4+1 and conclude that the missing number must be 1 less than 8. Or, one can decompose the 8 as 7+1, and perform the sequence of transformations 8+4=(7+1)+4=7+(1+4)=7+5. In later cases, students may develop a flexible decomposition of numbers, as they may be motivated by the equivalence of the quantities on both sides of the equation. All these aspects are essential to the development of relational thinking.

The teacher must underline the properties, used often implicitly by the students, when discussing these strategies. In each case, the students should be encouraged to clearly state the property of the operation on which those equivalences rely. It is suggested that the equations would be set up in a way that requires working with one specific property at time.

True or false (Molina & Mason, 2009)

Students should indicate if each equality is true or false and justify. It should be noted that direct calculation is not always a better strategy. For example, the expression 78-16 = 78-10-6 is true because on both sides we leave with the same number (78) and we remove the same value (16 = 10 + 6). This is the kind of strategy that needs to be promoted in students.

$72 = 56 - 14$
$78 - 16 = 78 - 10 - 6$
$24 - 15 = 24 - 10 - 5$
$78 - 45 = 77 - 44$
$100 + 94 - 94 = 100$
$27 - 14 + 14 = 26$
$231 + 48 = 231 + 40 + 8$
$13 - 5 + 5 = 13$
$51 + 51 = 50 + 52$
$15 - 6 = 6 - 15$
$27 - 14 + 14 = 26$
$93 = 93$
$24 - 24 = 0$

$7 + 7 + 9 = 14 + 9$
$10 - 7 = 10 - 4$
$7 + 3 = 10 + 3$
$62 - 13 + 13 = 65$
$19 - 3 = 18 - 2$
$13 + 11 = 12 + 12$
$10 + 4 = 4 + 10$
$0 + 325 = 326$
$37 + 22 = 300$
$125 - 0 = 125$
$7 = 12$
$100 - 100 = 1$

Describe the relationship (Blanton et al., 2015)

From the following situations, children are asked to describe the relationship between two given quantities.

1) The number of dogs and the number of dog noses.

2) The number of days and the number of coins in Sarah's piggybank if Sarah receives a coin per day from her grandmother.

3) The number of square tables and the number of people that can be seated at Brady's birthday if the tables are all juxtaposed and guests sit face-to-face at each table but not at the end of the table.

4) The number of people and the number of ears taking into consideration that each person has two ears.

5) The number of dogs and the number of dog legs knowing that each dog has 4 legs.

6) The size of a person and the height of that person when wearing a hat that is 1 foot in height.

7) The number of times you cut a piece of string placed in a straight line and the number of pieces of string obtained.

8) The number of candies of Marie and John have if they each have a box of candies containing the same number of candies and that Marie has 3 more on her box.

9) Initially Sarah has a piggy bank that contains coins and then she gets 3 additional coins. What is the number of coins in Sarah's piggy bank before and after adding the 3 coins?

10) The ages of Janice and Keisha if Janice is two years younger than Keisha.

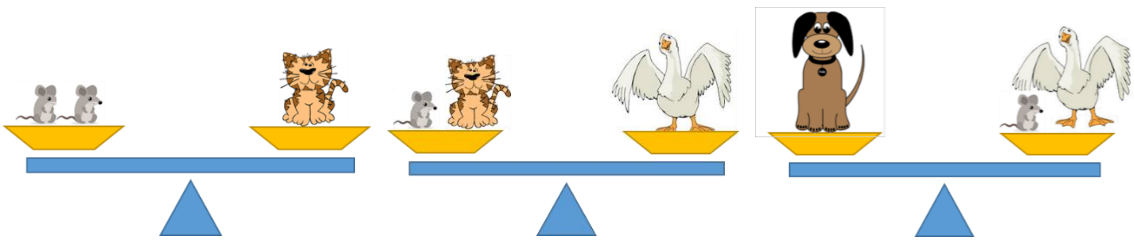
11) The length of a centipede (in number of segments that its body contains) and the number of days elapsed if each day the body of the thousand legs increases by 2 segments (we do not count the head).

Cycle 2 (8 to 10 years)

At this stage, students continue to analyze and generalize patterns, model relationships that gradually become more complex. Students may also be encouraged to think in a functional way in some particular contexts. Here are some typical activities supporting further development of students' algebraic thinking.

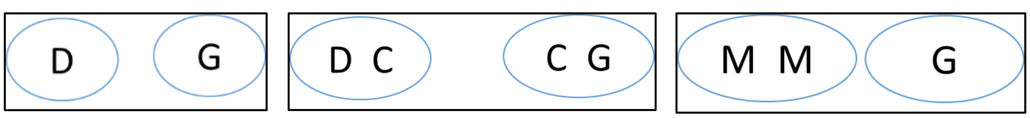
Iconic equations (Papadopoulos & Patsialia, 2018)

In this type of tasks, several equivalence relations are specified in a symbolic and pictorial way (not as formal equations), as illustrated below. (Ideas other than scales can be used: the strength of a team, the amount of food a group of animals can eat, etc.)

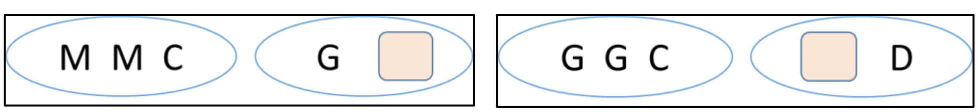


The students are then presented with three types of tasks: a) of compensation (finding the missing animal that would balance out); b) of identifying elements that hold the equivalence, and c) of letting students set up their own equivalences.

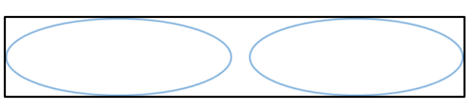
a) Add a mouse to make balance



b) Find the hidden animal



c) Create your own problem and propose it to your friend



The goal of these activities is to work with equivalence relations, by using substitution of some elements by another. Such a process is useful when solving equations, and students working in a familiar context can prepare them for later algebra work. The teacher should ask students to justify their choices by making references to the existing equivalences. The context has the advantage that students work with relations, rather than numbers – for finding a solution. Students must create equivalences instead of number facts. As with the use of letters, the teacher must be clear from the beginning what the pictures and icons represent. This might be a problematic aspect in certain situations, because students might think of the animals rather than some characteristic of it as being symbolized. In order to avoid confusion, one might use scales and specifically refer to the animal's weight. Such context might be more familiar, and students can accept more easily understand certain relations between weights, thus working with the given relations instead of relying on everyday experience and judgements about relative strength of animals.

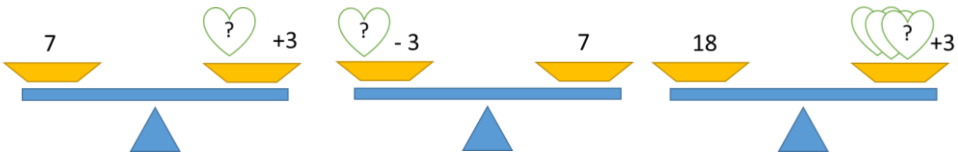
Idea of balance (Warren et al., 2009)

When dealing with equations and equivalences, an experience of working with balance is beneficial. Balance can model the equivalences expressed by the equal sign in formal representation. It is also possible to (visually) identify the consequences of modifications. For example, what happens when we take away only from one side or from both identical objects, thus supporting the process of solving to find the unknown.

While dealing with balance, the teacher must underline the *quantitative sameness* in a given situation. That is, that both sides are the same and information can be from either side (Warren & Cooper, 2005). In this sense, in order to reinforce the idea of equivalence, the teacher should employ more often the word "same" instead of "equal," because the latter has a procedural connotation. In this situation, it is important to display the three models: a concrete one (with the help of physical objects or a story), a pictorial one, and a symbolic one.

One limitation of the reference to the physical balance arrives in situations involving subtraction. By referring to physical balance, a certain relationship between quantities can be expressed in only one way. For example, the cat weighs as much as the mouse and the goose together—thus, exemplifying an idea of putting things together. Yet, the very same relation has other ways of being expressed. At certain stage of the development of mathematical thinking, the idea of balance can still be used as a **model** and not as a direct reference to physical balance. Below is an example of such use.

In this activity, images of scales are used to invite students reflect on possible operations, especially those that preserve equivalence. Students are asked to find "the weight" of the "unknown" object. Magnetic tapes can be used to build the scale model on the board and allow students to manipulate the magnetic tapes as they demonstrate their ideas. We can also compose a story for which the scale will be a model. For example, *Marjorie bought three hearts of chocolate and a bar that costs $3. She paid $18 in total.*

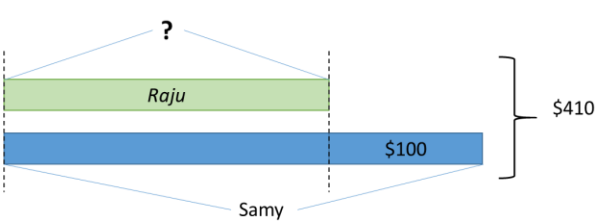


Problem solving (Cai et al, 2011; Polotskaia & Savard, 2018)

A different approach from *balance* is offered by the *model method,* also known as the Singapore method (see problem solving activity described in the Cycle 1 section above). In this approach, relations between quantities are translated into a pictorial equation, where quantities are modelled by segments or bars. The advantage of this method can be seen in the possibility to “read” the same relationship in three different ways (a=b+c; a-c=b; a-b=c). In this case, we remain in the same representational form, yet the interpretation of the relations differs. It is the reference quantity that changes—adding a new aspect to the representational flexibility. In problem solving both types of flexibility are required, moving forward and backward between different representational forms (equation, pictorial, concrete) as well as moving in the frame of the same representation in the particular way in which we express a relation (by choosing a reference quantity). The following task is an illustration on how this modelling method can be used in representing quantitative relations given in a particular context and in solving problems.

> *Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive? (*Cai et al., 2011, p. 2)

This problem, usually classified as an algebraic one, presents excellent opportunities to develop students’ relational and algebraic thinking without employing formal algebraic tools.



The role of the teacher in advancing and promoting the different modeling and representational forms cannot be emphasized enough. It falls back on the teacher to encourage students to translate representations from one form to another. Such transitions between representations should be done in every problem. Research demonstrated that linking parallel representations and coordinating at least two representation forms of a problem is an expression of mathematical comprehension (Duval, 2002), and as such is an essential component of algebraic thinking, as well as of problem solving in general.

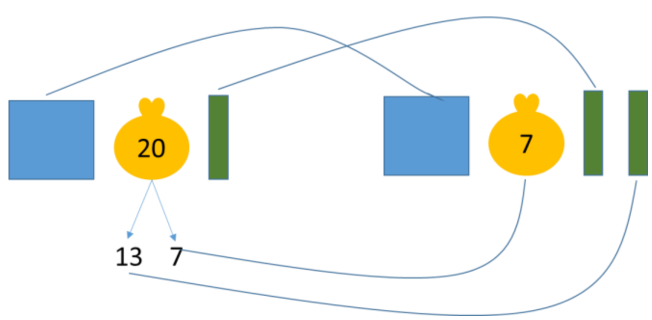
Problem solving (Brizuela & Schliemann, 2004)

Students are presented with the following problem:

> *Two students have the same amount of candies. Briana has one box, two tubes, and seven loose candies. Susan has one box, one tube, and 20 loose candies. If each box has the same amount [of candies] and each tube has the same amount [of candies], can you figure out how much [candy] each tube holds? What about each box? (p. 34)*

The situation itself is a comparison problem. The context is familiar and simple enough to engage students in solving it. Thus, as first step, the situation would be "acted-out" with physical objects. Under such circumstances, the student might use tacitly known rules of manipulation of equations. However, it is important for the teacher to suggest strategies for comparing: 1) "matching-up" identical amounts belonging to two different people; and 2) "cancelling out" (removing) identical amounts belonging to two different people. Verbalizing the strategies allows students to be aware of their implicit acts, by explicitly focusing on them. In this "acting-out" approach, students will decompose groups and will regroup them in order to simplify the situation up to a point when the problem is solved.

Yet, there should be a continuation of this activity with no physical objects involved in it. Instead, students should be encouraged to use pictorial/iconic representation of the situation with the students explicitly connecting identical elements from the two sides:



It is understood that this step prepares for a formal treatment of an equation.

It is up to the teacher to introduce the best way to represent the situation with letters. By specifying the meaning of the letters used, the teacher can ask students how they would formalize (write up mathematically) the involved quantities. Then, on this formal equation, the "matching-up" and "cancel-out" strategies can be lines up with the solution given in the pictorial representation. In this way, the formal treatment of the equation finds its roots in the physical manipulation and pictorial representation. The first two phases are essential for grounding the work for the symbolic solution.

Operations with holes (Carpenter et al., 2005)

At this stage, students work a lot with all four arithmetic operations and their properties, such as commutativity of multiplication, distributivity, the roles of "0" and "1" for multiplication etc. The tasks of "operations with holes" and "true or false," that we described in Cycle 1 section above, are useful tools in helping students understand these fundamental laws of mathematics. They can be easily modified to allow the development of this important knowledge. We present some examples. Note that they are not for a massive practice (worksheets), but each example should be used to attract students' curiosity and organize a classroom discussion.

$5 \times 2 - (5 \times \_\_) = 0$	$5 \div 2 - (\_\_ \div 2) = 0$
$5 \times 2 = (5 \times 5) - (5 \times \_\_)$	$10 \div 2 = (6 \div 2) + (4 \div \_\_)$
$(6 \times \_\_) = (2 \times 3) \times 4$	$24 \div \_\_ = (24 \div 3) \div 4$
$7 \times \_\_ = 7$	$7 \times \_\_ = 7$

The distributivity of multiplication on addition/subtraction offers many opportunities to develop relational thinking. The teacher should give students tasks that require the representation of relations between quantities in order for them to be solved. Thus, instead of a direct calculation (procedural approach), they are to use relations and properties to arrive to an answer.

For example, if a student is asked to calculate 4x9, but he knows the result only for 2x9, and applies it by considering 4x9 as 2x9+2x9, or formally, as 4x9=(2+2)x9, it would illustrate a relational approach rather than purely computational.

This type of activity also constitutes the context for working with the meaning of operations like in the above case with the meaning of multiplication. For example, if the task presented to the student is: 52x11 = (52x10) +*m*. What would *m* have to equal to make that a true number sentence? (Carpenter et al, 2005). The student should be encouraged to look for a solution by using the meaning of multiplication. We can also model numerical expressions by using schemas.

Thus, the teacher must sustain the student's reasoning by conducting a discussion in which the reference to the properties and meaning of operations is made explicit by a mathematical formalism. Overall, the non-standard equations used in this category of tasks should be set up in a way that the solution is more natural (easier, straightforward) by the use of the properties of operations than by calculations.

Patterns (Nacarato et al., 2017)

The "Form a train" task asks students to form a pattern with peers. Students form a line. Regularity is represented by student gestures and body positioning. For example: hand raised up / down; standing / kneeling, etc. The repeating element must be present in the line at least twice. A different student must identify the repeating element and position himself in the line according to this pattern. Wijns (2019) suggests that one can also encode the pattern using letters (ex. A B C A B C) or propose a code for the team to form by choosing gestures and positions.

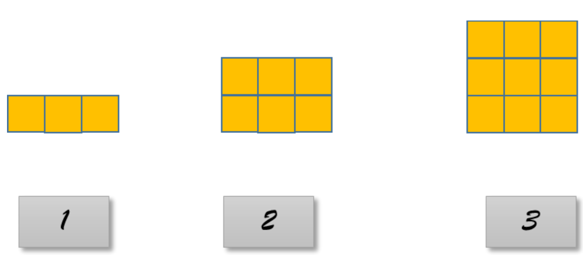
The researchers underscore the importance of allowing all students to propose hypotheses, discuss them, verify them, and defend them. The role of the teacher is to ensure that eventually all students are comfortable formulating (using mathematical language) and reuse the rule of each sequence that was discussed.

Articulating series and function-machine (Moss et al., 2011)

Note: *The principle is to go back and forth between the 3 tasks in order to establish connections. Students must be involved in the production of sequences and in proposing rules for the machine since this involvement is an important step for consolidation. The rules are expressed in words. We can also propose stories for which the sequence will be a model.*

Task 1: Relate a figure to its position in a sequence

Sample story: *The worker is building a wall of cement blocks. He installs three blocks per hour. Here are the "photos" taken after 1 hour, 2 hours and 3 hours.*



The photos form a sequence of patterns.

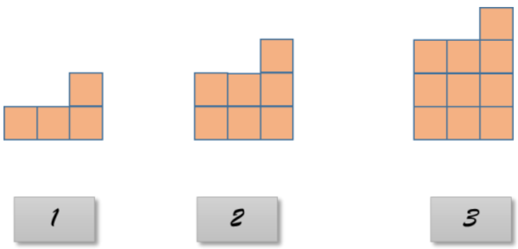
a) If we continue to build patterns in the same way, what will the pattern look like in the next position? How many blocks will this pattern include?

b) What would the pattern look like in the 10th position? How many blocks would this pattern include?

c) And what about the pattern at the 100th position?

d) And for any position? What could be the rule? (This question is asked after doing task 2.)

We ask the same questions with this new sequence:

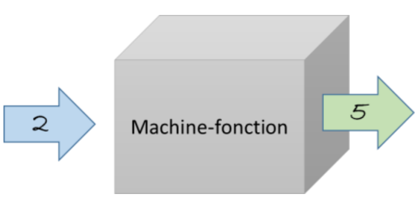


f) Construct another sequence of patterns with blocks. Ask your classmates to determine the rule of this sequence.

Task 2: "Guess my rule!"

A "machine" turns a number into another number according to a rule. Enter a number in the "In" slot and look at the number that comes out in the "Exit" slot. Guess what's the rule.

You can create a machine in your turn to play with your peers: choose a rule and when a number is entered in the machine you must return the number obtained after transformation. Your peers should guess the rule.

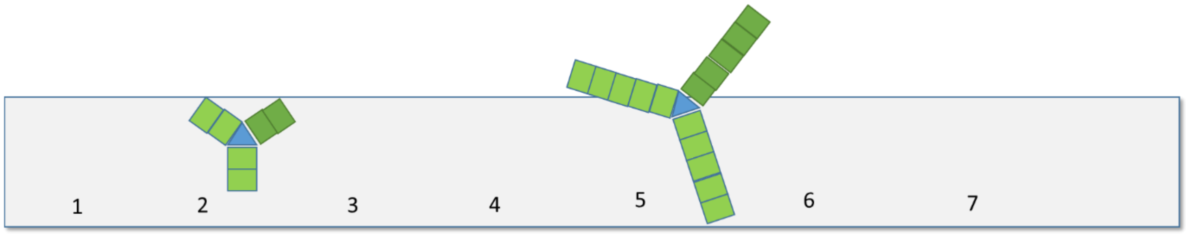


Input number	**Output number**
10	20
6	12
8	16

Our rule is	*Number times 2*

Task 3: The pattern sidewalk

A large paper tape is placed on the floor. Positions 1 to 10 are indicated.



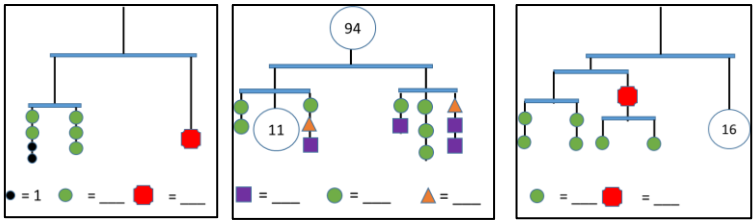
The teacher constructs a pattern at any position (the 5th for example) and then asks the students to advance a conjecture about the rule used for constructing the pattern according to its position. Then the teacher builds a second pattern that is not consecutive to the one specified before. The teacher asks the children to validate, or change, their rule, if necessary. The teacher builds a final pattern and then takes the students' proposals for the rule. Every child who makes a proposal must come and manipulate a construction in order to explain it.

Cycle 3 (10-12 years)

In Cycle 3, we continue the development of algebraic thought in increasingly complex contexts. These are the complex problems that contribute the most to the deepening and mastery of previously learned mathematical concepts. Exposure to complexity is a crucial and indispensable step in the mathematical education of students. Here we present some activities that promote students' formation of the components of algebraic thought so that they become tools of thought.

The concept of quantitative equivalence (Papadopoulos, 2019)

A series of interesting activities is proposed by Papadopoulos (2019). This involves presenting students with quantitative equivalences expressed by mobile images. Obviously, the mobile does not represent an image faithful to physical reality, but a model of equivalence. Working with this model requires a high level of abstraction. Each mobile is a combination of simple scales in equilibrium. Each object (see figure below) represents a "weight" so that the identical figures have the same values. The numbers in the circles represent known values, and a circle with a number at the head of the mobile represents its total weight. Combinations can have "equations" of varying degrees of complexity. Here are some examples of problems with mobiles.



Students are encouraged to use their knowledge of scale strategies to gradually simplify the presented situation and find the numerical values of the objects.

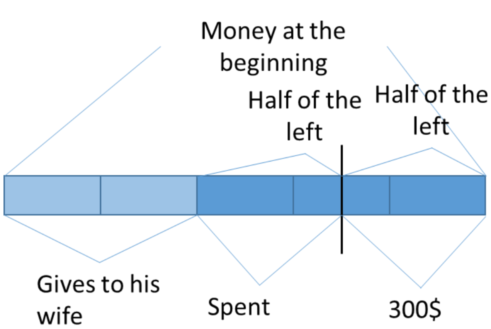
Every mobile problem can be expressed as a system of equations and solved by formal algebra. However, mobile problems are not tools for formalizing algebraic strategies. The ultimate goal of these activities is to deepen students' understanding of the idea of balance and quantitative equivalence, and to do this at an intermediate level of abstraction.

As in other mathematical activities, the teacher can invite students to verbalize the situation or compose a contextual problem that will correspond to it. Students are also asked to explain their solution strategy and demonstrate its validity.

Problem solving (Beckman, 2004)

The use of segment or strip diagrams allows students to work with very complex written problems however, without applying formal algebraic procedures. This way of working greatly favors the understanding of the mathematical relations and the laws associated with the four arithmetic operations. Here are some examples of problems and possible representations.

> *Ron gave 2/5 of his money to his wife and spent half of what was left. Finally, he has $ 300 left. How much money did he have at first?*

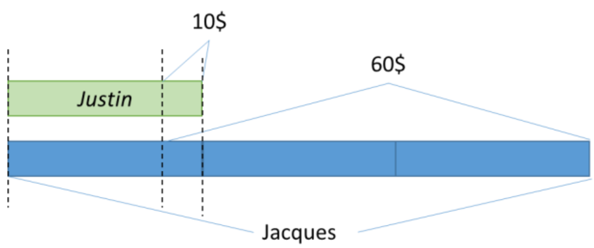


This model of the situation can be developed gradually from the statements of the problem's text. Once built, it allows proposing several strategies of calculation to find the unknown. For example, we can notice that the remaining amount of $ 300 represents 3 equal parts, two of which correspond to 1/5 of the starting amount. So, we can find this amount as $300 \div 3 \times 2 \times 5$. As variant, we can see the amount before spending the half as 300×2, and that corresponds to the 3/5 of the initial amount. Thus, we can proceed as follows: $300 \times 2 \div 3 \times 5$.

It is important to note that all computational strategies are visually supported on the model. Therefore, the model gives more meaning to the properties of arithmetic operations. In addition, these solutions can be obtained for any value of the remaining amount and are then general: M×2÷3×5.

Here is another example of a problem for which a solution without visual representation seems very difficult.

> *Jacques had 3 times more money than Justin. When Jacques spent $60 and Justin spent $10, it turned out they had the same amount. How much money did Jacques have at first?*



The visual analysis of the model of the problem leads us to note that $10 represents the difference between the $60 that Jacques spent and the amount included in 2/3 of his starting amount. In other words, the amount of $60 is composed of two equal parts and $10. From here, our calculation strategy will be: $(60 - 10) \div 2 \times 3$.

Beckman (2004) attributes the outstanding performance by Singaporean high school students in international TIMSS competitions to the widely used practice at primary level in Singapore of the written problem-solving practice supported by segment and band models.

« Functional » treatment of a word problem (inspired by Boyce & Moss, 2019; Carraher et al., 2005)

Several traditional written problems can be used to develop students' functional thinking. Here we present an example to demonstrate the transformation of the problem and clarify the didactic aspects of the solving activity.

> Typical problem: *Peter is trying to raise money to pay for a trip to Montreal. He already has $20 in his piggy bank. His parents give him $5 each day, if he helps them in the garden. How soon will Peter raise the $85 for the trip?*

Transformed problem: *Peter is trying to raise money to pay for a trip to Montreal. At first, he already had $20 in his piggy bank. His parents give him $5 each day if he helps them in the garden. In how many days will Peter raise the money for the trip?*

The purpose of transforming the problem is to find a formulation that prevents the student from starting a calculation immediately as we do not know the desired amount. Instead, students are asked to describe the process of accumulating money, to graph it on a Cartesian plane, and to match the number of days with the accumulated sum.

The situation, which now has become dynamic, can be described in two distinct ways:

- Each day we add $5: **recursive manner**.
- On any given day, the sum is composed of $20 plus $5 taken the number of times equal to the number of days: **functional manner**.

To discuss these two ways of thinking, one can also represent the situation in the form of a table of values. Students should be aware that the state of the piggy bank must be represented before the start of the day count.

Number of days	Accumulated sum
0	20
1	25
2	30
3	35
n	?

If we move in the right-hand side column only (+5, +5, ...), we need many "steps" to arrive at the desired sum. On the other hand, this reasoning shows us what changes from one day to another. A more efficient way to answer the question of the problem (even if we do not know the desired amount), is to find a direct dependence between the sum and the number of days. We therefore look for a formula (general calculation procedure) to calculate the number of days (D) knowing the sum (S) or vice versa: $D = (S - 20) \div 5 \; or \; S = 5 \times D + 20$.

This activity invites the student to use several methods of representation and modeling, to establish a correspondence between these models, to think in terms of "arbitrary" number and in terms of change or process, and to develop recursive and functional thinking.

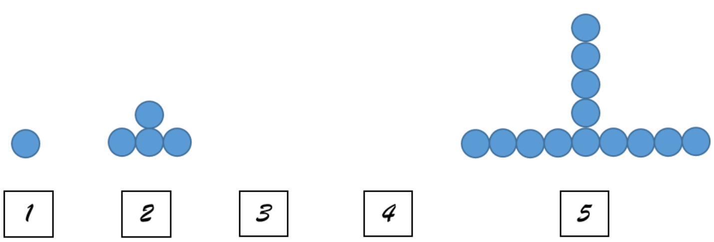
For the same problem context, the question can be formulated differently. If Peter's parents want his dream to come true within a certain number of days (say 31), how much do they have to give him a day? In this case, a functional relationship must be established between the desired sum (S) and the amount per day (A). $S = A \times 31 + 20$ or $A = (S - 20) \div 31$.

Figurative sequences and functional thinking (Blanton et al., 2015; Cooper & Warren, 2008; Rivera & Becker, 2011; Wilkie, 2019)

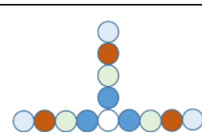
The figurative sequences are recognized by the research as an essential didactic tool in the development of the algebraic and, especially, functional thinking in students. The literature offers many examples of sequences and descriptions on how to treat them in the classroom. However, research suggests that certain didactical elements must be respected if it were to help all students develop this type of thinking.

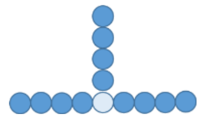
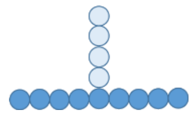
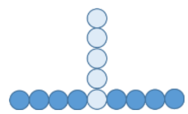
- To promote functional (rather than recursive) thinking, it is better to present the sequences with missing terms, rather than only the first terms one after the other.
- Before asking students to produce or describe the underlying functional relationship, we need to discuss with them the various ways of "seeing" the figures, their decomposition into simpler elements which, too, may be related to the position of the figure in the sequence.
- Each "viewing" of the figures can provoke a different reasoning and a different formulation of the sought relation. It is interesting and very enriching to compare the "viewings" and formulas obtained by students.

Here is an example of a figurative sequence (Wilkie & Clarke, 2016).



The following table contains different interpretations ("viewing") of the students with each leading to a different formula.

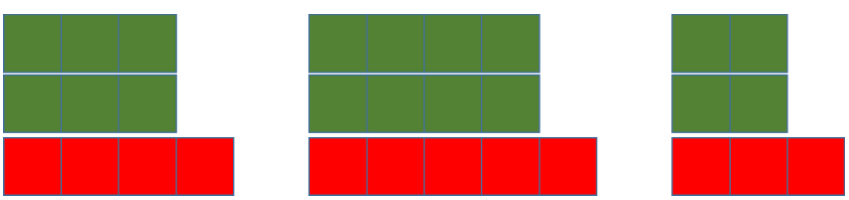
Description	**View(decomposition)**	**Formula**
We add 3 circles in the next figure		$1 + 3 + 3 + 3 \ldots$

The center with 3 identical segments		$1 + 3(n-1)$
A horizontal and a vertical line		$2n - 1 + (n-1)$
A vertical and two horizontal lines		$n + 2(n-1)$

To analyze the relation between the position of the figure and the number of circles that compose it, one must use various representations and notations including the manipulation with the physical material (tokens, geometrical figures, etc.). For example, one can put the data on known figures in a table, on a Cartesian plane or in formula form. In all cases, it is the responsibility of the teacher to ensure that each student understands the employed notation and representations, and that the student can confidently move from one representation to another without losing the sense of what this representation represents.

In the following example, the proposed situation is not in the form of a growing sequence. However, students were able to recognize the structure and arrange the figures in ascending order (Wilkie et Clarke, 2016).

Here is a collection of trucks built in a special way using red and green squares.



- *What is this special way? Describe it.*
- *If someone gives you several red squares, how many green squares will you need to build a truck from the same collection? How will you proceed to find this number of green squares?*
- *Can you describe the calculation rule using the letter R for the number of red squares and the letter G for the number of green squares?*

Increasing sequences can be composed starting from everyday situations. For example, Rivera and Becker (2011) propose to measure a stack of chairs, arranged one on top of another.

The height of the stack of chairs is composed of the height of the first chair and the heights added, one for each chair added. The researchers suggest that work on this task does not stop with the finding of a calculation formula. For example, students have found that to calculate the height of the pile in centimeters you must follow the procedure:

$$(number\ of\ chairs - 1) \times 7 + 80$$

So the teacher can suggest that students think about what they can say about chairs of another type, for which the formula is different:

$$(number\ of\ chairs - 1) \times 11 + 54$$

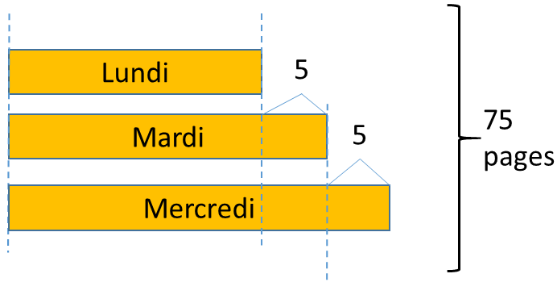
What is the height of each chair? (54) *What happens if we add a new chair?* (+11)

Complexity and flexibility of thinking (Heuvel-Panhuizen et al., 2013)

For algebraic thought to become a real tool for problem solving for the student, it is necessary to put the students in a situation where they must simultaneously use several structures and different ways of thinking. Here is a situation in this sense, proposed by Heuvel-Panhuizen et al. (2013).

> *There are 75 pages in a book. Petra starts reading on Monday. On Tuesday, she reads 5 pages more than Monday. On Wednesday, she reads 5 pages more than Tuesday and finishes the book. How many pages did Petra read on Wednesday?*

First, to understand this problem, one must imagine the process of increasing the number of pages read each day. This represents a **functional/recursive thought**. Subsequently, one can **model** the situation using the bar model. The model will capture the **relationship between all elements** and allow finding a computational strategy. These three modes of thinking—functional, relational, and modeling—are needed to unravel the situation and solve it. Such process of solution will be difficult for students to reach if they meet this challenge for the first time, and have no training in algebraic thought.



On the other hand, the combination of challenges will promote the mastery of each of these ways of thinking and will allow the creation of meaningful links between them.

Conclusion

We synthesize our conclusion in a form of a series of general recommendations. Their purpose is to improve the teaching of mathematics in elementary and kindergarten and to promote the development of algebraic thought among young people.

1. **Set up specific training for pre- and in-service teachers.**

Several studies noted that teachers do not have the skills and abilities needed to implement effective practices for the development of algebraic thought in kindergarten and primary levels. Indeed, they are not initiated into such approaches as students, nor in their initial training in teaching, mainly because research in algebraic though is relatively recent. Moreover, knowledge in formal algebra acquired at the secondary and college level is insufficient since it does not correspond to the contents of these new practices in early grades. We underline again that these new practices are, in fact, based on the implicit principles at the source of algebraic thinking.

2. Prioritize a relational approach to arithmetic.

Traditionally, arithmetic operations are treated as processes. For example, 2 + 3 = 5 is read from left to right and interpreted as "add 3 to 2 gives 5." This operational vision draws attention to the result obtained, that is 5. However, adopting a relational perspective implies taking into consideration the whole equation: 2 + 3 = 5 as it expresses the equivalence relation between two arithmetic expressions (2 + 3 and 5), thus 2 + 3 and 5 represent the same amount. This relational vision draws attention to the equivalence relation between 2 + 3 and 5.

The relational approach makes it possible to highlight the fundamental laws of arithmetic and algebra (example: the commutativity of addition, a + b = b + a). The study of these laws contributes positively to the construction of arithmetic as well as algebraic knowledge. Some authors propose to approach the study of these laws by the generalization of the student's (numerical) arithmetic experience. This approach is that of "algebra as generalized arithmetic." For example, the student may notice that 2 + 3 = 3 + 2, that 5 + 7 = 7 + 5, that 12 + 9 = 9 + 12, and so on, thus, to induce that it is probably true for all numbers.

Others propose to introduce these laws first to the students in the context of qualitative comparison of physical objects (length, volume, weight, etc.). Only then, these laws are exploited in arithmetic (numerical) and algebraic contexts. This last approach can be identified as "algebra at the base of arithmetic." For example, the commutativity observed on two juxtaposed lengths (strips of paper) and generalized from the beginning by means of the mathematical language (A + B = B + A) is easily transferred to the cases of addition of natural numbers or algebraic terms.

Therefore, 5+6=6+5 and 2x+3y=3y+2x.

3. Introduce early the use of letters as representing quantities.

The literal notation is very useful for expressing and communicating relations and fundamental laws in a general, concentrated, and easily observable form. It has been shown that the use of letters in mathematical communication is not an obstacle, and students from the age of 5-6 can benefit from learning to generalize and develop increasingly abstract and complex mathematical ideas. Researchers (ex. Davydov, 2008; Lee, 2006; Hewitt, 2012) explain that the use of letters in mathematics is a tool developed within the mathematical culture and children can learn it, like all other cultural tools, through immersion. Students must be exposed to the uses and constraints of the use of letters, and be encouraged to use them in their reflections and mathematical communications.

Note that the use of letters in mathematics is varied. The letter can mean a constant quantity, known or unknown, a variable quantity, or a general number.

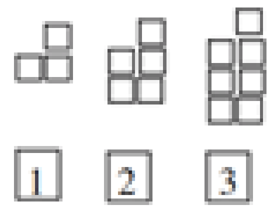
4. Promote the study of complex situations.

Several researchers propose that only problem solving with complex quantitative relationships ensures the development of deep and flexible mathematical reasoning. A complex problem (with multiple relationships between quantities) requires more sophisticated analysis and more elaborate planning than a simple problem (presenting a single relationship). Faced with a complex problem, students develop both the flexibility to choose and use acquired thought tools, and new cognitive and metacognitive tools.

Moreover, it is in this context that the fundamental laws of mathematics, the relational approach as well as the various strategies of reasoning make sense and become essential, and useful, tools for the student.

5. Enhance students' familiarity with co-varying quantities.

To prepare students for the study of high school functions, several scholars propose the use of tasks in which the student must analyze the dependence between two quantities that can take several values. Example: the study of an increasing series of figures in which the number of elements of each figure depends on the position of the figure in the series and vice versa.



Other authors propose to simply add a reflection on covariation starting from traditional written problems. For example, from the following problem: *Anna has 5 pencils and Marta has 3 pencils more than Anna. How many pencils does Marta have?* We can engage students' thinking about the covariation of the number of Anna's and Marta's pencils. For example, we can ask the following questions: *If Anna has 2 pencils, what can we say about the number of pencils in Marta? What if Anna has 10 pencils? What if Anna has x pencils, how can one express the number of Marta's pencils?* The authors emphasize that the understanding of the notion of "variable," which is necessary at the secondary level, develops through situations in which one is interested in varying quantities. Thus, it is important to introduce students to this type of situation from primary school.

6. Introduce and constantly use modeling as a way of thinking.

Modeling is recognized by researchers as a tool of reasoning and mathematical generalization. Researchers (ex. Corral, 2019; Davydov, 2008; Mason, 2018) point out that, from the point of view of learning, the use of representations is effective if the representation plays the role of model of the object or situation studied. Example: if students analyze a repetitive series of geometric blocks (square, circle, circle, square, circle, circle, etc.) the teacher can propose to model the sequence as ABBABBABB and then ask the students to build other examples that match this pattern (e.g., red, green, green, red, green, green; or large, small, small, large, small, small). Usually, what is represented by the model is the set of essential relationships that determines the mathematical meaning of the object or situation. The practice of modeling then favors the generalization and acquisition of the mathematical relations and fundamental laws of arithmetic and algebra.

7. **Emphasize the learning of specific representations to boost the use of modeling and generalization.**

Representations, such as the number line, Cartesian plane, tables, Arrange-All schematization, the use of letters and mathematical symbolism, etc., are tools of reasoning and mathematical modeling. Researchers find that simply using varied representations in mathematics learning is not enough. Indeed, to ensure the access of all students to the generalization and development of more and more abstract ideas, the use and understanding of certain representations is essential. However, each representation tool must be used in alignment with the purpose of learning.

For example, a representation by tens and units facilitates the calculation and learning of the number system. On the other hand, in the case of solving a written problem, we first look for the operation or operations to be performed. Therefore, in this case, to analyze the relationships between quantities, an Arrange-All representation is more relevant. Systems of specific representations need to be introduced at times when the learning content requires it and when the students feel the need so to facilitate their mathematical communication and thinking. In addition, researchers emphasize the importance of varying the tools of representations as well as mathematical thought patterns to equip students with a set of strategies to be used flexibly rather than favoring one particular strategy over others. Therefore, problem solving should not be limited to the use of algebraic strategies. Arithmetic or other strategies are sometimes more effective.

8. **Highlight mathematical discussion in class.**

All researchers insist on adopting a culture of mathematical discussion in the classroom. This culture includes, among other things, the use of an adequate mathematical language, the valorization of the reasoning of each student (even if this reasoning is, sometimes, not clear, is not complete or is not correct). Each student is invited to propose their vision of the situation, to formulate a hypothesis, to share their strategy and elaborate on it by way of supporting evidence. Each opinion should be discussed, justified, and clarified for all students by means of generating discussions. The time used for mathematical discussions is not lost but invested in the development of students' sense of ownership over mathematical ideas and their development as mathematical thinkers. Within such culture, mathematical error is an opportunity for learning rather than a problem or an obstacle. In several experiments, students were asked to analyze errors made by a fictional character of mathematically impossible situations or incorrect mathematical communications. The students then had the opportunity to engage in discussions about the nature of errors made and identify misperceptions or misconceptions.

All these didactic tools favor the development of students' mathematical reasoning and contribute positively to their personal and social development as users and doers of mathematics.

9. Vary teaching according to the nature of the content taught.

In the experiments reported and consulted for the purpose of this synthesis, the teaching methods used by researchers vary according to the purpose of the activity and the different contents discussed. For example, during the pattern sequence activities, students needed an explanation of how to analyze the patterns visually and what to look for in order to clarify the pattern. This required explicit teaching of these skills. Using this new skills, students analyzed several sequences (patterns) to discover recursive and functional rules in an almost autonomous way, which is in line with the principles of problem-based teaching that was found to ensure effective learning. On the other hand, students were constantly encouraged and expected to respect the mathematical culture by justifying their ideas and discussing the strategies proposed by their peers, which exemplifies immersion teaching.

These recommendations are the result of our synthesis of knowledge drawn from the work of researchers around the world. This knowledge confirms that algebraic thought is within the reach of young students and that the main question is about the choice of didactic materials and teaching strategies. However, a lot of work needs to be done to build a curriculum tailored to the needs of students that will truly ensure the reduction of arithmetic-algebra gup.

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